Asset Pricing with Heterogeneous Belief and Preference under Portfolio Constraints

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Abstract

I study the interaction between portfolio constraints and incentives to trade. I construct a general equilibrium model in continuous time so the equilibrium conditions can be explicitly characterized. The model has two agents disagreeing about the growth rate of the dividend of a real asset. The agents also have different risk aversion parameters. Each agent faces a short sale constraint and a borrowing constraint. Furthermore, the borrowing constraint depends on the market value of the real asset. I characterize conditions under which each constraint is binding. The short sale constraint, when binding, decreases the equity premium and the market price of risk of the real asset. The borrowing constraint, when binding, increases the equity premium and the market price of risk of the real asset. The long-run probability density of the model states features two modes concentrating on either end of the state space, so both agents have positive wealth in the long run.

1 Introduction

How does belief disagreement lead to different binding portfolio constraints when the investment opportunity sets are ex ante identical? How is belief disagreement different from differences in risk aversion or patience? I use an endowment economy to show that, among others, the most significant result of belief disagreement is that the short sale motive is so strong that the short sale constraint binds in equilibrium.

I construct a model with two agents disagreeing about the stochastic process of the dividend of a real asset. Each agent faces a short sale constraint and a borrowing constraint. Furthermore, the borrowing constraint is contingent on the market value of the real asset, so it features the "balance sheet effect" emphasized by Kiyotaki and Moore (1997). I show that the short sale constraint, when binding, decreases the equity premium and the market price of risk of the real asset; and the borrowing constraint, when binding, increases the equity premium and the market price of risk of the real asset. Furthermore, I derive the long-run probability density of the model states, which features two modes concentrating on either ends of the state space.

To answer my question, it is important that both investors face short sale and borrowing constraints ex ante. This is a challenging computation task. My solution technique is closely related to Chabakauri (2015) and Rytchkov (2014). Instead of letting only one investor face only one type of constraint, which is the common practice in the literature, I allow both investors face both types of constraints. Technically, my formulation makes the problem with occasionally binding constraints harder to solve. More importantly, it makes the investors face identical investment opportunity sets ex ante. With the more general setting, I am able to study the effects of short sale constraints and borrowing constraints when they could simultaneously show up in the model. This allows for potential interaction between the two constraints, which is absent in most papers due to the exclusive assumptions. However, I show analytically that there is at most one constraint binding in any given state. Furthermore, the numerical exercise shows that the expectation of another constraint binding in the future states has no effect on the current constraint. That is, when the short sale constraint binds my model looks the same as a model with only short sale constraint, and similarly for the borrowing constraint. These results justify the exercise of exclusively assuming one type of constraint for one agent.

2 Economic environment

There is one risky asset that yields a continuous flow of non-storable output (dividend) that can be consumed. The flow rate of dividend D_t evolves according to the following diffusion process:

$$\mathrm{d}D_t = D_t \left(\mu_D \mathrm{d}t + \sigma_D \mathrm{d}z_t\right)$$

where μ_D and σ_D are exogenous parameters. σ_D is known by all investors while μ_D is unknown. There are two investors A and B, who observe D_t but hold different beliefs about the drift of the dividend process. More formally, they have their own probability spaces $\left\{\Omega, \left\{\mathcal{F}_t^i\right\}, \mathbb{P}^i\right\}$ with *subjective probability measures* $\mathbb{P}^i$, which are equivalent with the true probability measure $\mathbb{P}$. The dynamics of the dividend process under investor A and B's beliefs are

$$\mathrm{d}D_t = D_t \left(\mu_D^i \mathrm{d}t + \sigma_D \mathrm{d}z_t^i\right), i \in \{A, B\},$$

where z^i is a standard Brownian motion under investor i's measure $\mathbb{P}^i$. Since the investors observe the dividends D_t, they should agree on $\mathrm{d}D_t/D_t$, hence $\mu_D \mathrm{d}t + \sigma_D \mathrm{d}z_t = \mu_D^A \mathrm{d}t + \sigma_D \mathrm{d}z_t^A = \mu_D^B \mathrm{d}t + \sigma_D \mathrm{d}z_t^B$, and

$$\mathrm{d}z_t^B = \mathrm{d}z_t^A - \Delta_D \mathrm{d}t, \tag{2.1}$$

$$\mathrm{d}z_t = \mathrm{d}z_t^i + \frac{\mu_D^i - \mu_D}{\sigma_D} \mathrm{d}t, i \in \{A, B\} \tag{2.2}$$

where $\Delta_D = \left(\mu_D^B - \mu_D^A\right)/\sigma_D$ is the measure of disagreement. I assume that $\Delta_D > 0$, so investor B is more optimistic about the dividend growth. For simplicity, investors don't update their beliefs.

There is a riskless asset whose value B_t has the following dynamics:

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t$$

where the initial bond price is normalized to $B_0 = 1$. There is also a risky asset (stock) whose price follows the process

$$\begin{aligned} \mathrm{d}S_t + D_t \mathrm{d}t &= S_t \left(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}z_t\right) \\ &= S_t \left(\mu_t^i \mathrm{d}t + \sigma_t \mathrm{d}z_t^i\right), i \in \{A, B\}. \end{aligned}$$

The two equation imply that

$$\mathrm{d}z_t^B = \mathrm{d}z_t^A - \frac{\mu_t^B - \mu_t^A}{\sigma_t} \mathrm{d}t,$$

$$\mathrm{d}z_t = \mathrm{d}z_t^i + \frac{\mu_t^i - \mu_t}{\sigma_t} \mathrm{d}t, i \in \{A, B\},$$

hence

$$\frac{\mu_t^B - \mu_t^A}{\sigma_t} = \frac{\mu_D^B - \mu_D^A}{\sigma_D} = \Delta_D,$$

$$\frac{\mu_t^i - \mu_t}{\sigma_t} = \frac{\mu_D^i - \mu_D}{\sigma_D}, i \in \{A, B\} \tag{2.3}$$

The investors face constraints on their portfolios, which will be specified later. The market is otherwise complete since the only source of uncertainty is the Brownian motion driving the dividend process and there is a riskless asset and one risky asset.

The parameters for the dividend process $\mu_D, \sigma_D, \mu_D^A, \mu_D^B$ are exogenous, whereas the parameters for the price processes r_t, μ_t and σ_t are to be determined endogenously by equilibrium conditions.

3 Investor's optimization problems

The investor's maximization problem is:

$$\max \mathbb{E}^i \left[\int_0^\infty e^{-\rho t} u_i \left(c_{it} \right) \mathrm{d}t \right]$$

$$\begin{aligned} \text{s.t. } \mathrm{d}W_{it} &= W_{it} \left(\theta_{it} \left(\mu_t^i - r_t \right) + r_t \right) \mathrm{d}t - c_{it} \mathrm{d}t + W_{it} \theta_{it} \sigma_t \mathrm{d}z_t^i \\ \underline{\theta} &\leq \theta_{it} \leq \frac{\bar{m}}{\sigma_t}. \end{aligned}$$

The first inequality is a constraint on short sales. Due to belief disagreement, short sale constraint is binding (Chabakauri (2015)) for the pessimistic investor (investor A). The second inequality can be derived from a Kiyotaki-Moore type collateral constraint. Suppose the investor invests in $h_1(t)$ shares of stocks and $h_2(t)$ shares of bonds, then the value of the portfolio is $W(t) = h_1(t) S(t) + h_2(t) B(t)$. Suppose $h_2(t) < 0$ so the investor borrows from the bond market to finance the investment in stocks. Suppose further that the investor has limited commitment so the bond has to be collateralized by the stock. At $t + \Delta$, the investor should repay $-h_2(t) B_t (1 + r_t \Delta)$, and the collateral value is $h_1(t) (S_{t+\Delta} + D_t \Delta)$. The investor will default if the collateral value is less than the amount of debt, hence the probability of default is less than b if

$$\Pr \left(h_1(t) \left(S_{t+\Delta} + D_t \Delta \right) + h_2(t) B_t \left(1 + r_t \Delta \right) \leq 0 \right) \leq b.$$

Notice that the portfolio value at $t + \Delta$ can be written as

$$\begin{aligned} h_1(t) \left(S_{t+\Delta} + D_t \Delta \right) + h_2(t) B_t \left(1 + r_t \Delta \right) &= \left[h_1(t) \left(S_{t+\Delta} - S_t + D_t \Delta \right) + h_2(t) B_t r_t \Delta \right] \\ &\quad + h_1(t) S_t + h_2(t) B_t \\ &= h_1(t) S_t \left(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}z_t \right) + h_2(t) B_t r_t \mathrm{d}t + W_t \\ &= W_t \left[\theta_t \left(\mu_t - r_t \right) \mathrm{d}t + r_t \mathrm{d}t + \theta_t \sigma_t \mathrm{d}z_t \right] + W_t \end{aligned}$$

as $\Delta \to 0$. To a first-order approximation, the probability can be written as

$$\Pr \left(\mathrm{d}z_t \leq -\frac{1}{\theta_t \sigma_t} \right) \leq b.$$

Let $\bar{m} = -1 / \left(\Phi^{-1}(b) \sqrt{\mathrm{d}t} \right)$, we have

$$\theta \leq \frac{\bar{m}}{\sigma_t}.$$

Notice that the volatility of stock price σ_t is state-dependent, so the portfolio constraint is also varying over time. In equilibrium, the constraint $\underline{\theta} \leq \theta_{it}$ is binding for investor A while the constraint $\theta_{it} \leq \frac{\bar{m}}{\sigma_t}$ is binding for investor B.

Let the Lagrange multiplier on the inequality constraint be λ_t and define

$$\nu_{it}^* = \frac{\lambda_{it}^*}{W_{it}\frac{\partial J}{\partial W}}$$

as the rescaled multiplier at the optimum. The asset price dynamics of the fictitious economy can be derived from the HJB equation:

$$\begin{aligned} 0 &= \max_{c_{it},\theta_{it}} \left\{ e^{-\rho t} u_i(c_{it}) + \frac{1}{\mathrm{d}t}\mathbb{E}_t^i[\mathrm{d}J] + \underline{\nu}_{it}^*[\theta_{it} - \underline{\theta}] W_{it}\frac{\partial J}{\partial W} + \bar{\nu}_{it}^*[\bar{m} - \sigma_t\theta_{it}] W_{it}\frac{\partial J}{\partial W} \right\} \\ &= \max_{c_{it},\theta_{it}} \left\{ e^{-\rho t} u_i(c_{it}) + \frac{\partial J}{\partial t} + \left[W_{it}\left(\theta_{it}\left(\mu_t^i + \underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t - r_t\right) + r_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}\right) - c_{it}\right]\frac{\partial J}{\partial W} \right. \\ &\left. + y_t\mu_{yt}\frac{\partial J}{\partial y} + \frac{1}{2}\left(W_{it}^2\theta_{it}^2\sigma_t^2\frac{\partial^2 J}{\partial W^2} + y_t^2\sigma_{yt}^2\frac{\partial^2 J}{\partial y^2}\right) + W_{it}\theta_{it}y_t\sigma_{yt}\sigma_t\frac{\partial^2 J}{\partial s\partial W} \right\}. \end{aligned}$$

Collecting terms and comparing with the standard HJB equation for an unconstrained economy, we find that the drift term for the riskless asset is $r_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}$, the drift term for the risky asset is $\mu_t^i + \underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}$ and the diffusion term is σ_t.

In the fictitious unconstrained economy, the bond and stock prices follow the adjusted dynamics

$$\mathrm{d}B_t = B_t\left(r_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}\right)\mathrm{d}_t,$$

$$\mathrm{d}S_t + D_t\mathrm{d}_t = S_t\left[\mu_t^i + \underline{\nu}_{it}^*(1-\underline{\theta}) - \bar{\nu}_{it}^*(\sigma_t - \bar{m})\right]\mathrm{d}t + S_t\sigma_t\mathrm{d}z_t^i$$

for investors A and B. Given the adjustment $\underline{\nu}_{it}^*$ and $\bar{\nu}_{it}^*$, the investor's problem can be solved as if in an unconstrained complete-market economy. The stochastic discount factors have the following dynamics:

$$\mathrm{d}\xi_{it} = -\xi_{it}\left[\left(r_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}\right)\mathrm{d}_t + \left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right)\mathrm{d}z_t^i\right] \tag{3.1}$$

where $\kappa_t^i = \left(\mu_t^i - r_t\right)/\sigma_t$ is the perceived market price of risk by investor i. The first-order condition requires that

$$e^{-\rho t}\left(c_{it}^*\right)^{-\gamma_i} = \psi_i\xi_{it},$$

hence $c_{it}^* = \left(e^{\rho t}\psi_i\xi_{it}\right)^{-1/\gamma_i}$. Market clearing condition requires that $c_{At}^* + c_{Bt}^* = D_t$. Applying Ito's lemma on both sides and matching the drift and diffusion terms gives the equilibrium processes for κ and r in terms of investor B's consumption share y and adjustment ν_{it}^*. Lemma 1 summarizes the results.

Lemma 1. *(Equilibrium process in terms of shadow costs of constraints) Given ν_{At}^* and ν_{Bt}^*, the market price of risk $\kappa_t = (\mu_t - r_t)/\sigma_t$, interest rate r_t, volatility σ_{yt} and drift μ_{yt} of consumption share $y_t = c_{Bt}^*/D_t$,* *and volatility of stock returns σ_t are functions of y_t, given by:*

$$\kappa_t = \Gamma_t \sigma_D + \frac{\mu_D - \bar{\mu}_{Dt}}{\sigma_D} - \Gamma_t \frac{1-y_t}{\gamma_A} \frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \Gamma_t \frac{y_t}{\gamma_B} \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t}, \tag{3.2}$$

$$r_t = \Gamma_t \Bigg\{ \mu_D^B + \frac{\rho_A (1-y_t)}{\gamma_A} + \frac{\rho_B y_t}{\gamma_B} + \frac{1-y_t}{\gamma_A} \left[\nu^*_{At}\underline{\theta} - \bar{\nu}^*_{At}\bar{m} - \Delta_D \left(\kappa_t + \frac{\mu_D^A - \mu_D}{\sigma_D} + \frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} \right) \right] - \frac{y_t}{\gamma_B} \left(-\underline{\nu}^*_{Bt}\underline{\theta} + \bar{\nu}^*_{Bt}\bar{m} \right)$$
$$- \frac{1}{2} \left[\frac{1+\gamma_A}{\gamma_A^2} \left(\kappa_t + \frac{\mu_D^A - \mu_D}{\sigma_D} + \frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} \right)^2 (1-y_t) + \frac{1+\gamma_B}{\gamma_B^2} \left(\kappa_t + \frac{\mu_D^B - \mu_D}{\sigma_D} + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right)^2 y_t \right] \Bigg\}, \tag{3.3}$$

$$\sigma_{yt} = \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B)\sigma_D + \Delta_D - \left(\frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right) \right], \tag{3.4}$$

$$\mu_{yt} = \frac{r_t - \underline{\nu}^*_{Bt}\underline{\theta} + \nu^*_{Bt}\bar{m} - \rho_B}{\gamma_B} - \mu_D - \sigma_D \sigma_{yt} + \frac{\mu_D^B - \mu_D}{\sigma_D} (\sigma_{yt} + \sigma_D) + \frac{1+\gamma_B}{2} (\sigma_{yt} + \sigma_D)^2, \tag{3.5}$$

$$\sigma_t = \frac{\Psi'(y_t)}{\Psi(y_t)} y_t \sigma_{yt} + \sigma_D. \tag{3.6}$$

where

$$\Gamma_t = \frac{\gamma_A \gamma_B}{(1-y_t)\gamma_B + y_t \gamma_A}, \text{ and } \bar{\mu}_{Dt} = \Gamma_t \left(\frac{1-y_t}{\gamma_A} \mu_D^A + \frac{y_t}{\gamma_B} \mu_D^B \right).$$

Proposition 2. *(Equilibrium HJB equation) Suppose there exists a Markov equilibrium with wealth-consumption ratios $\Phi_i(y) \in C^1[0,1] \cap C^2(0,1)$. Then, the stock price-dividend ratio $\Psi(y) = (1-y)\Phi_A(y) + y\Phi_B(y)$, and investor i's value function $J_i(W_{it}, y_t, t)$ and optimal portfolio weight θ_i^* are given by*

$$J_i(W_{it}, y_t, t) = e^{-\rho_i t} \frac{W_{it}^{1-\gamma_i} \Phi_i(y_t)^{\gamma_i}}{1-\gamma_i} \tag{3.7}$$

and

$$\theta^*_{it} = \frac{1}{\gamma_i \sigma_t} \left[\left(\kappa_t^i + \frac{\underline{\nu}^*_{it} - \bar{\nu}^*_{it}\sigma_t}{\sigma_t} \right) + \gamma_i y_t \sigma_{yt} \frac{\Phi_i'(y_t)}{\Phi_i(y_t)} \right]. \tag{3.8}$$

The wealth-consumption ratio $\Phi_i(y_t)$ of investor i satisfies the second-order ordinary differential equation (ODE):

$$0 = \frac{y_t^2 \sigma_{yt}^2}{2} \Phi_i'' + y_t \left[\frac{(1-\gamma_i)}{\gamma_i} \sigma_{yt} \left(\kappa_t^i + \frac{\underline{\nu}^*_{it} - \bar{\nu}^*_{it}\sigma_t}{\sigma_t} \right) + \mu^i_{yt} \right] \Phi_i'$$
$$+ \left(-\rho_i + (1-\gamma_i) \left(\frac{1}{2\gamma_i} \left(\kappa_t^i + \frac{\underline{\nu}^*_{it} - \bar{\nu}^*_{it}\sigma_t}{\sigma_t} \right)^2 + r_t - \underline{\nu}^*_{it}\underline{\theta} + \bar{\nu}^*_{it}\bar{m} \right) \right) \frac{\Phi_i}{\gamma_i} + 1. \tag{3.9}$$

The boundary values are given by

$$\Phi_A(0) = \frac{\gamma_A}{\rho_A - (1-\gamma_A) \left(\frac{1}{2\gamma_A} \left(\kappa_t^A + \frac{\underline{\nu}^*_{At}(0) - \bar{\nu}^*_{At}(0)\sigma_D}{\sigma_D} \right)^2 + r_t(0) - \underline{\nu}^*_{At}(0)\underline{\theta} + \bar{\nu}^*_{At}(0)\bar{m} \right)}$$

$$\Phi_A(1) = \frac{\gamma_A}{\rho_A - (1-\gamma_A) \left(\frac{1}{2\gamma_A} \left(\kappa_t^A + \frac{\underline{\nu}^*_{At}(1) - \bar{\nu}^*_{At}(1)\sigma_t(1)}{\sigma_t(1)} \right)^2 + r_t(1) - \underline{\nu}^*_{At}(1)\underline{\theta} + \bar{\nu}^*_{At}(1)\bar{m} \right)}$$

$$\Phi_B(0) = \frac{\gamma_B}{\rho_B - (1-\gamma_B)\left(\frac{1}{2\gamma_B}\left(\kappa_t^B + \frac{\underline{\nu}_{Bt}^*(0) - \bar{\nu}_{Bt}^*(0)\sigma_D}{\sigma_D}\right)^2 + r_t(0) - \underline{\nu}_{Bt}^*(0)\underline{\theta} + \bar{\nu}_{Bt}^*(0)\bar{m}\right)}$$

$$\Phi_B(1) = \frac{\gamma_B}{\rho_B - (1-\gamma_B)\left(\frac{1}{2\gamma_B}\left(\kappa_t^B + \frac{\underline{\nu}_{Bt}^*(1) - \bar{\nu}_{Bt}^*(1)\sigma_t(1)}{\sigma_t(1)}\right)^2 + r_t(1) - \underline{\nu}_{Bt}^*(1)\underline{\theta} + \bar{\nu}_{Bt}^*(1)\bar{m}\right)}$$

where $r_t(0) =, r_t(1)$, $\underline{\nu}_{it}^*(0), \underline{\nu}_{it}^*(1)$, $\bar{\nu}_{it}^*(0)$, *and* $\bar{\nu}_{it}^*(1)$ *denote their values when* $y = 0$ *or* 1. *Finally, at most one of the four multipliers could be strictly positive in any state, and their values are given by*

$$\underline{\nu}_{At}^* = \begin{cases} \frac{\sigma_D[(\gamma_A-\gamma_B)\sigma_D+\Delta_D]}{(1-\underline{\theta})g_{1t}^A+\underline{\theta}} + \frac{(1-\underline{\theta})\sigma_D^2 g_{1t}^A}{g_{2t}^A((1-\underline{\theta})g_{1t}^A+\underline{\theta})^2} & \text{if } \underline{\theta}_{At}^* = \underline{\theta}, \underline{\nu}_{At}^*\underline{\theta} < 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\underline{\nu}_{Bt}^* = \begin{cases} -\frac{\sigma_D[(\gamma_A-\gamma_B)\sigma_D+\Delta_D]}{(1-\underline{\theta})g_{1t}^A+\underline{\theta}} - \frac{(1-\underline{\theta})\sigma_D^2 g_{1t}^B}{g_{2t}^B((1-\underline{\theta})g_{1t}^B+\underline{\theta})^2} & \text{if } \underline{\theta}_{Bt}^* = \underline{\theta}, \underline{\nu}_{Bt}^*\underline{\theta} < 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\nu}_{At}^* = \begin{cases} \left(1 - \frac{g_{3t}^A}{\bar{m}}\right)\frac{\bar{m}}{g_{2t}^A} & \text{if } \bar{\theta}_{At}^* = \frac{\bar{m}}{\sigma_t}, \bar{\nu}_{At}^*\bar{m} > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\nu}_{Bt}^* = \begin{cases} \left(\frac{g_{3t}^B}{\bar{m}} - 1\right)\frac{\bar{m}}{g_{2t}^B} & \text{if } \bar{\theta}_{Bt}^* = \frac{\bar{m}}{\sigma_t}, \bar{\nu}_{Bt}^*\bar{m} > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$g_{1t}^A = \frac{-1 + (1-y_t)\frac{\Phi_A'}{\Phi_A}}{-1 + (1-y_t)\left(\frac{\Phi_A'}{\Phi_A} - \frac{\Psi'}{\Psi}\right)}, g_{2t}^A = \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(-1 + (1-y_t)\frac{\Phi_A'}{\Phi_A}\right), g_{3t}^A = \sigma_D + ((\gamma_A-\gamma_B)\sigma_D + \Delta_D)\, g_{2t}^A,$$

and

$$g_{1t}^B = \frac{1 + y_t\frac{\Phi_B'}{\Phi_B}}{1 + y_t\left(\frac{\Phi_B'}{\Phi_B} - \frac{\Psi'}{\Psi}\right)}, g_{2t}^B = \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi_B'}{\Phi_B}\right), g_{3t}^B = \sigma_D + ((\gamma_A-\gamma_B)\sigma_D + \Delta_D)\, g_{2t}^B.$$

That is, at most one investor is constrained in any state.

4 Equilibrium

I compute two equilibria under the following set of parameter values: $\mu_D = 0.018, \sigma_D = 0.032, \rho_A = \rho_B = 0.01, m = 1.5\sigma_D$. In the model with belief disagreement but homogeneous risk aversion, I set $\mu_D^A = 0.65\mu_D^B = 0.65\mu_D, \gamma_A = \gamma_B = 2$. In the model with the same belief but heterogeneous risk aversion, I set $\mu_D^A = \mu_D^B = \mu_D, \gamma_A = 10, \gamma_B = 2$.

I compute the distribution using Kolmogorov forward equation:

$$\frac{\partial}{\partial t}p(t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma_y^2 p(t,y)\right) - \frac{\partial}{\partial y}\left(\mu_y p(t,y)\right)$$

where σ_y and μ_y are the diffusion and drift terms of the process of y_t: $\mathrm{d}y_t = y_t(\mu_y \mathrm{d}t + \sigma_y \mathrm{d}z_t)$. $p(t,y)$ is the pdf of the state variable at time t, with the initial condition suppressed. The stationary distribution is

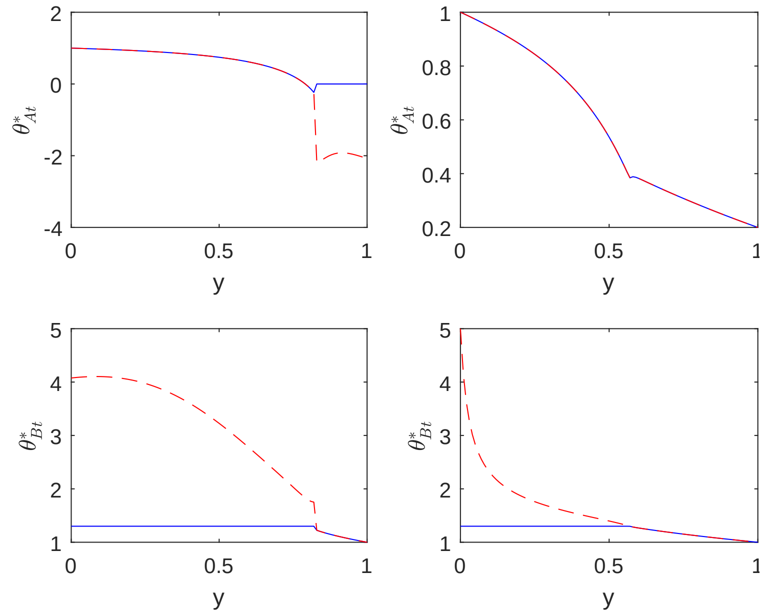


Figure 4.1: Equilibrium portfolio weight on the risk asset. First column: $\mu_D^A = 0.65\mu_D^B, \gamma_A = \gamma_B = 2$. Second column: $\mu_D^A = \mu_D^B, \gamma_A = 10, \gamma_B = 2$. Blue : constrained economy; red: unconstrained economy.

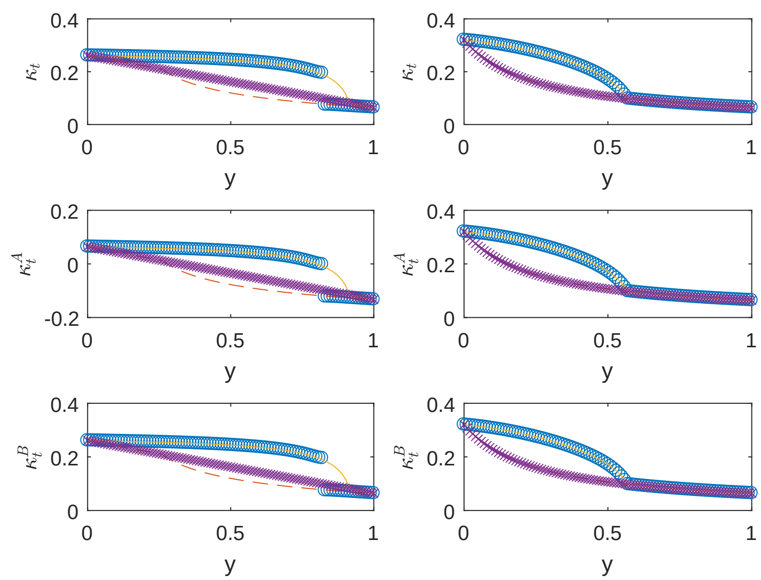


Figure 4.2: Equilibrium market price of risk. First column: $\mu_D^A = 0.65\mu_D^B, \gamma_A = \gamma_B = 2$. Second column: $\mu_D^A = \mu_D^B, \gamma_A = 10, \gamma_B = 2$. $\circ$: both constraints; –: borrowing constraint only; - -: short sale constraint only; $\times$: no constraint.

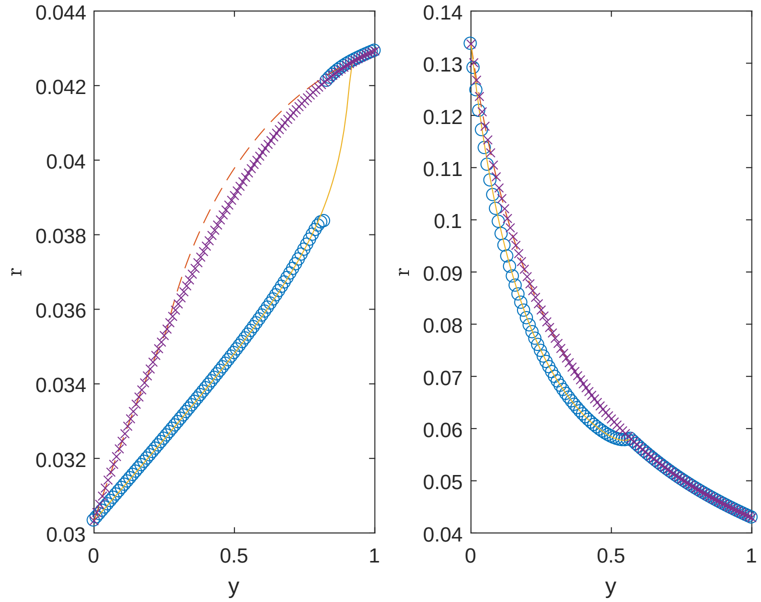


Figure 4.3: Equilibrium risk-free rate. First column: $\mu_D^A = 0.65\mu_D^B, \gamma_A = \gamma_B = 2$. Second column: $\mu_D^A = \mu_D^B, \gamma_A = 10, \gamma_B = 2$. $\circ$: both constraints; –: borrowing constraint only; - -: short sale constraint only; $\times$: no constraint.

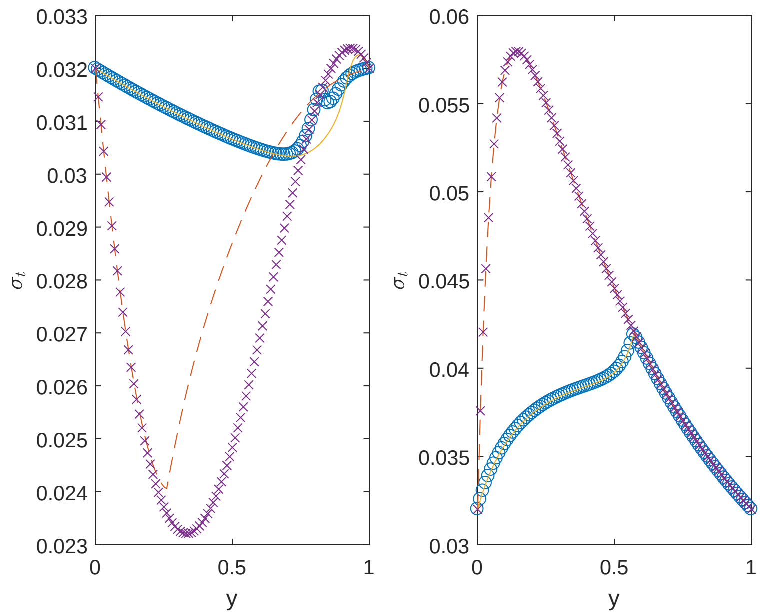


Figure 4.4: Equilibrium volatility of the return on risky asset. First column: $\mu_D^A = 0.65\mu_D^B, \gamma_A = \gamma_B = 2$. Second column: $\mu_D^A = \mu_D^B, \gamma_A = 10, \gamma_B = 2$. $\circ$: both constraints; –: borrowing constraint only; - -: short sale constraint only; $\times$: no constraint.

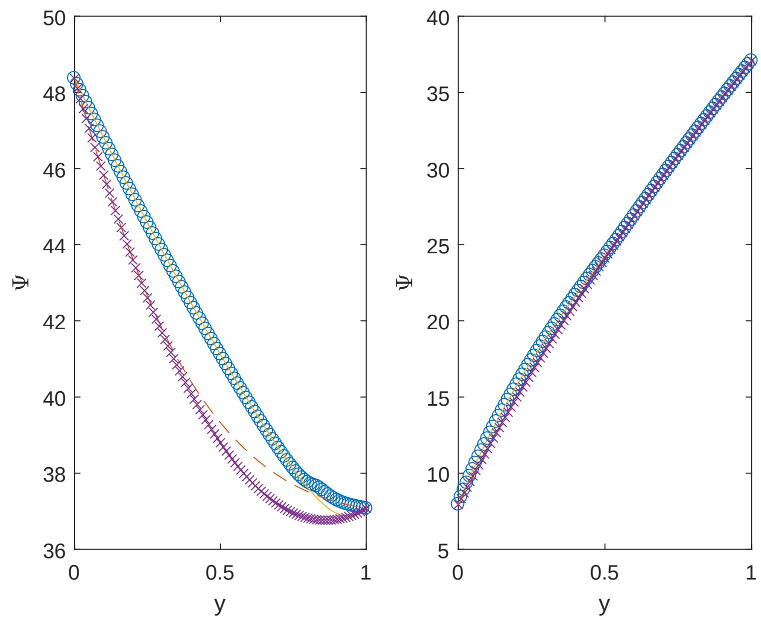


Figure 4.5: Equilibrium P/D ratio. First column: $\mu_D^A = 0.65\mu_D^B, \gamma_A = \gamma_B = 2$. Second column: $\mu_D^A = \mu_D^B, \gamma_A = 10, \gamma_B = 2$. $\circ$: both constraints; –: borrowing constraint only; - -: short sale constraint only; $\times$: no constraint.

computed by setting the time derivative $\partial p/\partial t = 0$. Letting $D(y) = \sigma_y^2 p(y)$, the ODE for the stationary distribution is

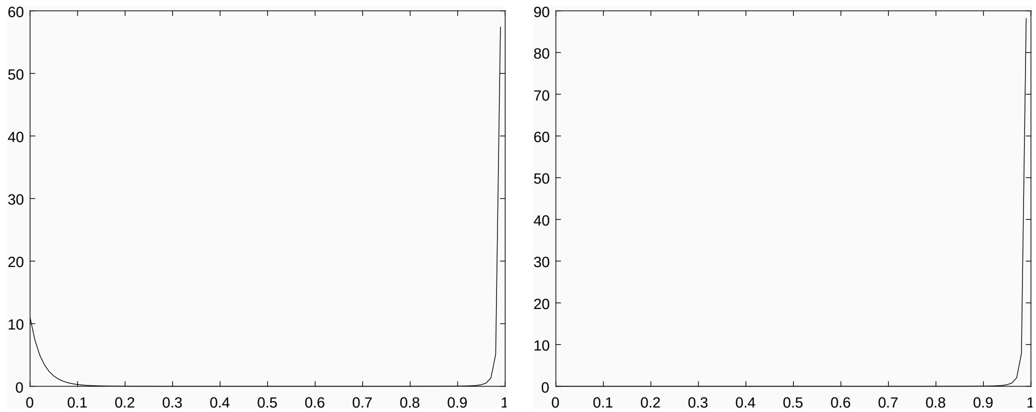
$$D'(y) = 2\frac{\mu_y}{\sigma_y} D(y),$$

and $d(y) = D(y)/\sigma_y^2$. The initial condition is such that $\int_0^1 d(y)\,\mathrm{d}y = 1$.

Figure 4.6 panel (a) plots the stationary distribution for the model with heterogeneous belief and heterogeneous risk aversion. Interestingly, the stationary distribution is bi-modal. In a typical two-agent model, the more aggressive investor holds all wealth in the stationary distribution because she holds more risky asset in her portfolio and the risky asset earns higher expected return than the risk-free asset. It is not the case in this model. The less aggressive investor also holds non-zero wealth in the stationary distribution. Borrowing constraint is the key to the survival of the less aggressive investor. Under the borrowing constraint, the more aggressive investor is unable to earn high return on the risky asset when the constraint binds, so her wealth can't always outgrow the wealth of the less aggressive investor.

5 Conclusion

I analyze effects of portfolio constraints on asset prices and long-run wealth distribution in a two-agent endowment economy. The two investors differ by their beliefs about endowment growth and risk aversion. Both investors face borrowing constraint and short sale constraint. In equilibrium, belief heterogeneity is crucial for the short sale constraint to bind. The model is able to generate bi-modal stationary distribution of wealth when short sale constraint is in place.



(a) Both constraints. (b) Short sale constraint only.

Figure 4.6: Stationary distribution for the economy with heterogeneous belief.

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Appendix

Proof of Lemma 1. First, using equation (2.1), we could rewrite the dynamics of ξ_{At} in equation (3.1) as:

$$\mathrm{d}\xi_{At} = -\xi_{At}\left[\left(r_t - \underline{\nu}_{At}^*\underline{\theta} + \bar{\nu}_{At}^*\bar{m} + \Delta_D\left(\kappa_t^A + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)\right)\mathrm{d}t + \left(\kappa_t^A + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)\mathrm{d}z_t^B\right]$$

Substitute the optimality condition $c_{it}^* = \left(e^{\rho t}\psi_i\xi_{it}\right)^{-1/\gamma_i}$ into the market clearing condition $c_{At}^* + c_{Bt}^* = D_t$, applying Ito's lemma to both sides, and dividing by D_t:

$$\begin{aligned}
\mu_D^B \mathrm{d}t + \sigma_D \mathrm{d}z_t^B \quad = \quad & \left\{ -\frac{\rho_A}{\gamma_A}\mathrm{d}t + \frac{1}{\gamma_A}\left[\left(r_t - \underline{\nu}_{At}^*\underline{\theta} + \bar{\nu}_{At}^*\bar{m} + \Delta_D\left(\kappa_t^A + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)\right)\mathrm{d}t + \left(\kappa_t^A + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)\mathrm{d}z_t^B\right]\right.\\
& \left. + \frac{1}{2}\frac{1+\gamma_A}{\gamma_A^2}\left(\kappa_t^A + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)^2 \mathrm{d}t\right\}(1-y_t)\\
+ \quad & \left\{ -\frac{\rho_B}{\gamma_B}\mathrm{d}t + \frac{1}{\gamma_B}\left[\left(r_t - \underline{\nu}_{Bt}^*\underline{\theta} + \bar{\nu}_{Bt}^*\bar{m}\right)\mathrm{d}t + \left(\kappa_t^B + \frac{\underline{\nu}_{Bt}^* - \bar{\nu}_{Bt}^*\sigma_t}{\sigma_t}\right)\mathrm{d}z_t^B\right] + \frac{1}{2}\frac{1+\gamma_B}{\gamma_B^2}\left(\kappa_t^B + \frac{\underline{\nu}_{Bt}^* - \bar{\nu}_{Bt}^*\sigma_t}{\sigma_t}\right)^2 \mathrm{d}t\right\} y_t.
\end{aligned}$$

Matching the drift and diffusion terms on both sides:

$$\begin{aligned}
\mu_D^B = & -\frac{\rho_A(1-y_t)}{\gamma_A} - \frac{\rho_B y_t}{\gamma_B} \qquad (5.1)\\
& + \frac{1-y_t}{\gamma_A}\left[r_t - \underline{\nu}_{At}^*\underline{\theta} + \bar{\nu}_{At}^*\bar{m} + \Delta_D\left(\kappa_t^A + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)\right] + \frac{y_t}{\gamma_B}\left(r_t - \underline{\nu}_{Bt}^*\underline{\theta} + \bar{\nu}_{Bt}^*\bar{m}\right)\\
& + \frac{1}{2}\left[\frac{1+\gamma_A}{\gamma_A^2}\left(\kappa_t^A + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)^2(1-y_t) + \frac{1+\gamma_B}{\gamma_B^2}\left(\kappa_t^B + \frac{\underline{\nu}_{Bt}^* - \bar{\nu}_{Bt}^*\sigma_t}{\sigma_t}\right)^2 y_t\right],
\end{aligned}$$

$$\sigma_D = \frac{1-y_t}{\gamma_A}\left(\kappa_t^A + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right) + \frac{y_t}{\gamma_B}\left(\kappa_t^B + \frac{\underline{\nu}_{Bt}^* - \bar{\nu}_{Bt}^*\sigma_t}{\sigma_t}\right). \qquad (5.2)$$

Next, by construction we have $\kappa_t^i - \kappa_t = \left(\mu_t^i - \mu_t\right)/\sigma_t = \left(\mu_D^i - \mu_D\right)/\sigma_D$, where the second equality follows from equation (2.3). Therefore $\kappa_t^i = \kappa_t + \left(\mu_D^i - \mu_D\right)/\sigma_D$. Substituting into equations (5.1) and (5.2):

$$\begin{aligned}
\mu_D^B = & -\frac{\rho_A(1-y_t)}{\gamma_A} - \frac{\rho_B y_t}{\gamma_B}\\
& + \frac{1-y_t}{\gamma_A}\left(r_t - \underline{\nu}_{At}^*\underline{\theta} + \bar{\nu}_{At}^*\bar{m} + \Delta_D\left(\kappa_t + \frac{\mu_D^A - \mu_D}{\sigma_D} + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)\right) + \frac{y_t}{\gamma_B}\left(\left(r_t - \underline{\nu}_{Bt}^*\underline{\theta} + \bar{\nu}_{Bt}^*\bar{m}\right)\right)\\
& + \frac{1}{2}\left[\frac{1+\gamma_A}{\gamma_A^2}\left(\kappa_t + \frac{\mu_D^A - \mu_D}{\sigma_D} + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right)^2(1-y_t) + \frac{1+\gamma_B}{\gamma_B^2}\left(\kappa_t + \frac{\mu_D^B - \mu_D}{\sigma_D} + \frac{\underline{\nu}_{Bt}^* - \bar{\nu}_{Bt}^*\sigma_t}{\sigma_t}\right)^2 y_t\right],
\end{aligned}$$

$$\sigma_D = \frac{1-y_t}{\gamma_A}\left(\kappa_t + \frac{\mu_D^A - \mu_D}{\sigma_D} + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right) + \frac{y_t}{\gamma_B}\left(\kappa_t + \frac{\mu_D^B - \mu_D}{\sigma_D} + \frac{\underline{\nu}_{Bt}^* - \bar{\nu}_{Bt}^*\sigma_t}{\sigma_t}\right).$$

κ_t and r_t can be solved from the above system of equations:

$$\begin{aligned}
\kappa_t = & \frac{\gamma_A\gamma_B}{(1-y_t)\gamma_B + y_t\gamma_A}\left[\sigma_D - \frac{1-y_t}{\gamma_A}\left(\frac{\mu_D^A - \mu_D}{\sigma_D} + \frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t}\right) - \frac{y_t}{\gamma_B}\left(\frac{\mu_D^B - \mu_D}{\sigma_D} + \frac{\underline{\nu}_{Bt}^* - \bar{\nu}_{Bt}^*\sigma_t}{\sigma_t}\right)\right]\\
= & \Gamma_t\sigma_D + \frac{\mu_D - \bar{\mu}_{Dt}}{\sigma_D} - \Gamma_t\frac{1-y_t}{\gamma_A}\frac{\underline{\nu}_{At}^* - \bar{\nu}_{At}^*\sigma_t}{\sigma_t} - \Gamma_t\frac{y_t}{\gamma_B}\frac{\underline{\nu}_{Bt}^* - \bar{\nu}_{Bt}^*\sigma_t}{\sigma_t},
\end{aligned}$$

$$r_t = \Gamma_t \left\{ \mu_D^B + \frac{\rho_A (1-y_t)}{\gamma_A} + \frac{\rho_B y_t}{\gamma_B} + \frac{1-y_t}{\gamma_A} \left[\underline{\nu}^*_{At}\underline{\theta} - \bar{\nu}^*_{At}\bar{m} - \Delta_D \left(\kappa_t + \frac{\mu_D^A - \mu_D}{\sigma_D} + \frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} \right) \right] - \frac{y_t}{\gamma_B} \left(-\underline{\nu}^*_{Bt}\underline{\theta} + \bar{\nu}^*_{Bt}\bar{m} \right) \right.$$
$$\left. - \frac{1}{2} \left[\frac{1+\gamma_A}{\gamma_A^2} \left(\kappa_t + \frac{\mu_D^A - \mu_D}{\sigma_D} + \frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} \right)^2 (1-y_t) + \frac{1+\gamma_B}{\gamma_B^2} \left(\kappa_t + \frac{\mu_D^B - \mu_D}{\sigma_D} + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right)^2 y_t \right] \right\}$$

where

$$\Gamma_t = \frac{\gamma_A \gamma_B}{(1-y_t)\gamma_B + y_t \gamma_A}, \text{ and } \bar{\mu}_{Dt} = \Gamma_t \left(\frac{1-y_t}{\gamma_A} \mu_D^A + \frac{y_t}{\gamma_B} \mu_D^B \right).$$

The volatility σ_{yt} and drift μ_{yt} of the consumption share $y_t = c^*_{Bt}/D_t$ can be obtained by applying Ito's lemma to the ratio c^*_{Bt}/D_t:

$$\frac{\mathrm{d}y_t}{y_t} = \frac{\mathrm{d}c^*_{Bt}}{c^*_{Bt}} - \frac{\mathrm{d}D_t}{D_t} + \left(\frac{\mathrm{d}D_t}{D_t} \right)^2 - \frac{\mathrm{d}c^*_{Bt}}{c^*_{Bt}} \frac{\mathrm{d}D_t}{D_t}$$
$$= -\frac{\rho_B}{\gamma_B}\mathrm{d}t + \frac{1}{\gamma_B} \left[\left(r_t - \underline{\nu}^*_{Bt}\underline{\theta} + \bar{\nu}^*_{Bt}\bar{m} \right) \mathrm{d}t + \left(\kappa_t^B + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right) \frac{\mu_D^B - \mu_D}{\sigma_D} \mathrm{d}t + \left(\kappa_t^B + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right) \mathrm{d}z_t \right]$$
$$+ \frac{1}{2} \frac{1+\gamma_B}{\gamma_B^2} \left(\kappa_t^B + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right)^2 \mathrm{d}t - (\mu_D \mathrm{d}t + \sigma_D \mathrm{d}z_t) + \sigma_D^2 \mathrm{d}t - \sigma_D \frac{\left(\kappa_t^B + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right)}{\gamma_B} \mathrm{d}t.$$

Collecting the drift and diffusion terms,

$$\sigma_{yt} = \frac{\kappa_t^B + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t}}{\gamma_B} - \sigma_D$$
$$= \frac{\kappa_t + \frac{\mu_D^B - \mu_D}{\sigma_D} + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t}}{\gamma_B} - \sigma_D$$
$$= \frac{\Gamma_t \sigma_D + \frac{\mu_D - \bar{\mu}_{Dt}}{\sigma_D} - \Gamma_t \frac{1-y_t}{\gamma_A} \frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \Gamma_t \frac{y_t}{\gamma_B} \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} + \frac{\mu_D^B - \mu_D}{\sigma_D} + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t}}{\gamma_B} - \sigma_D$$
$$= \frac{\Gamma_t \sigma_D + \frac{(1-y_t)\gamma_B}{(1-y_t)\gamma_B + y_t\gamma_A}\Delta_D - \Gamma_t \frac{1-y_t}{\gamma_A} \frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \Gamma_t \frac{y_t}{\gamma_B} \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t}}{\gamma_B} - \sigma_D$$
$$= \frac{(1-y_t)(\gamma_A - \gamma_B)\sigma_D + (1-y_t)\Delta_D - (1-y_t)\left(\frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right)}{(1-y_t)\gamma_B + y_t\gamma_A}$$
$$= \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} \left[(\gamma_A - \gamma_B)\sigma_D + \Delta_D - \left(\frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right) \right].$$

$$\mu_{yt} = -\frac{\rho_B}{\gamma_B} + \frac{1}{\gamma_B} \left[r_t - \underline{\nu}^*_{Bt}\underline{\theta} + \bar{\nu}^*_{Bt}\bar{m} + \left(\kappa_t^B + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right) \frac{(\mu_D^B - \mu_D)}{\sigma_D} \right] + \frac{1}{2} \frac{1+\gamma_B}{\gamma_B^2} \left(\kappa_t^B + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right)^2$$
$$- \mu_D + \sigma_D^2 - \frac{\sigma_D \left(\kappa_t^B + \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} \right)}{\gamma_B}$$
$$= \frac{r_t - \underline{\nu}^*_{Bt}\underline{\theta} + \nu^*_{Bt}\bar{m} - \rho_B}{\gamma_B} + \frac{\mu_D^B - \mu_D}{\sigma_D}(\sigma_{yt} + \sigma_D) + \frac{1+\gamma_B}{2}(\sigma_{yt} + \sigma_D)^2 - \mu_D + \sigma_D^2 - \sigma_D(\sigma_{yt} + \sigma_D)$$
$$= \frac{r_t - \underline{\nu}^*_{Bt}\underline{\theta} + \nu^*_{Bt}\bar{m} - \rho_B}{\gamma_B} - \mu_D - \sigma_D\sigma_{yt} + \frac{\mu_D^B - \mu_D}{\sigma_D}(\sigma_{yt} + \sigma_D) + \frac{1+\gamma_B}{2}(\sigma_{yt} + \sigma_D)^2.$$

Finally, the volatility of stock returns σ_t can be derived from the fact that $S_t = \Psi(y_t) D_t$ where $\Psi(y_t) = S_t/D_t$. Applying Ito's lemma:

$$\begin{aligned}\frac{\mathrm{d}S_t}{S_t} &= \frac{\Psi'(y_t)}{\Psi(y_t)}\mathrm{d}y_t + \frac{\mathrm{d}D_t}{D_t} + \frac{1}{2}\left[\frac{\Psi''(y_t)}{\Psi(y_t)}(\mathrm{d}y_t)^2 + 2\Psi'(y_t)\frac{\mathrm{d}D_t}{D_t}\frac{\mathrm{d}y_t}{y_t}\right] \\ &= \left(\frac{\Psi'(y_t)}{\Psi(y_t)}y_t\mu_{yt} + D_t\mu_D + \frac{1}{2}\frac{\Psi''(y_t)}{\Psi(y_t)}y_t^2\sigma_y^2 + \frac{\Psi'(y_t)}{\Psi(y_t)}D_t y_t\sigma_D\sigma_{yt}\right)\mathrm{d}t \\ &+ \left(\frac{\Psi'(y_t)}{\Psi(y_t)}y_t\sigma_{yt} + \sigma_D\right)\mathrm{d}z_t.\end{aligned}$$

Therefore the volatility of stock return is

$$\sigma_t = \frac{\Psi'(y_t)}{\Psi(y_t)}y_t\sigma_{yt} + \sigma_D.$$

Proof of Proposition 2. Let $\underline{\nu}_{it}^* W_{it}\partial J/\partial W$ and $\bar{\nu}_{it}^* W_{it}\partial J/\partial W$ denote the Lagrange multipliers associated with the lower and upper bounds on the portfolio weights of the stock. Investor i's Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned}0 &= \max_{c_{it},\theta_{it}}\left\{e^{-\rho_i t}u_i(c_{it}) + \frac{1}{\mathrm{d}t}\mathbb{E}_t^i[\mathrm{d}J] + \underline{\nu}_{it}^*[\theta_{it} - \underline{\theta}]W_{it}\frac{\partial J}{\partial W} + \bar{\nu}_{it}^*[\bar{m} - \sigma_t\theta_{it}]W_{it}\frac{\partial J}{\partial W}\right\} \\ &= \max_{c_{it},\theta_{it}}\left\{e^{-\rho_i t}u_i(c_{it}) + \frac{\partial J}{\partial t} + \left[W_{it}\left(\theta_{it}\left(\mu_t^i + \underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t - r_t\right) + r_t - \nu_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}\right) - c_{it}\right]\frac{\partial J}{\partial W}\right. \\ &\left. + y_t\mu_{yt}^i\frac{\partial J}{\partial y} + \frac{1}{2}\left(W_{it}^2\theta_{it}^2\sigma_t^2\frac{\partial^2 J}{\partial W^2} + y_t^2\sigma_{yt}^2\frac{\partial^2 J}{\partial y^2}\right) + W_{it}\theta_{it}y_t\sigma_{yt}\sigma_t\frac{\partial^2 J}{\partial W\partial y}\right\}\end{aligned}$$

subject to the transversality condition $\lim_{T\to\infty}\mathbb{E}_t[J] = 0$. The value function given by Equation (3.7) satisfies the HJB equation if and only if wealth-consumption ratios Φ_i satisfy Equations (3.9).

The optimal portfolio weight can be found by the first-order condition

$$W_{it}\left(\mu_t^i + \underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t - r_t\right)J_W + W_{it}^2\theta_{it}^*\sigma_t^2 J_{WW} + W_{it}y_t\sigma_{yt}\sigma_t J_{Wy} = 0$$

$$\begin{aligned}\theta_{it}^* &= \frac{\left(\mu_t^i + \underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t - r_t\right) + y_t\sigma_{yt}\sigma_t\gamma_i\Phi_i(y_t)^{-1}\Phi_i'(y_t)}{\gamma_i\sigma_t^2} \\ &= \frac{1}{\gamma_i\sigma_t}\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t} + \gamma_i y_t\sigma_{yt}\frac{\Phi_i'(y_t)}{\Phi_i(y_t)}\right).\end{aligned}$$

It is also useful to define the unconstrained optimal portfolio weight θ_{it}^{u*}:

$$\theta_{it}^{u*} = \frac{1}{\gamma_i\sigma_t}\left(\kappa_t^i + \gamma_i y_t\sigma_{yt}\frac{\Phi_i'(y_t)}{\Phi_i(y_t)}\right).$$

Note that the parameters $\sigma_t, \kappa_t^i,$ and σ_{yt} are computed by setting $\underline{\nu}_{it}^*$ and $\bar{\nu}_{it}^*$ equal to zero whereas $\underline{\nu}_{jt}^*$ and $\bar{\nu}_{jt}^*$ for the other investor remain unchanged. In words, θ_{it}^{u*} denotes the portfolio weight for investor i when she was the only one unconstrained, taking the other investor being constrained and the subsequent price processes as given. The first-order condition for consumption is

$$e^{-\rho_i t}\left(c_{it}^*\right)^{-\gamma_A} - J_W = 0$$

$$c_{it}^* = \frac{W_{it}}{\Phi_i(y_t)}.$$

Plugging the optimal consumption c_{it}^* and portfolio weight θ_{it}^* back into the HJB equation, we find the ODE for $\Phi_i(y_t)$:

$$\begin{aligned}
0 &= e^{-\rho_i t}\frac{\left(\frac{W_{it}}{\Phi_i}\right)^{1-\gamma_i}}{1-\gamma_i} - \rho_i e^{-\rho_i t}\frac{W_{it}^{1-\gamma_i}\Phi_i^{\gamma_i}}{1-\gamma_i} \\
&+ \left[W_{it}\left(\theta_{it}^*\left(\mu_t^i + \underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t - r_t\right) + r_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}\right) - \frac{W_{it}}{\Phi_i}\right] e^{-\rho_i t} W_{it}^{-\gamma_i}\Phi_i^{\gamma_i} \\
&+ y_t\mu_{yt}^i e^{-\rho_i t}\frac{W_{it}^{1-\gamma_i}\gamma_i\Phi_i^{\gamma_i-1}\Phi_i'}{1-\gamma_i} + \frac{1}{2}\Bigg\{W_{it}^2\left(\theta_{it}^*\right)^2\sigma_t^2 e^{-\rho_i t}\left(-\gamma_i\right)W_{it}^{-\gamma_i-1}\Phi_i^{\gamma_i} \\
&+ y_t^2\sigma_{yt}^2 e^{-\rho_i t}\frac{W_{it}^{1-\gamma_i}\gamma_i\left[\left(\gamma_i-1\right)\Phi_i^{\gamma_i-2}\left(\Phi_i'\right)^2 + \Phi_i^{\gamma_i-1}\Phi_i''\right]}{1-\gamma_i}\Bigg\} \\
&+ W_{it}\theta_{it}^* y_t\sigma_{yt}\sigma_t e^{-\rho_i t}W_{it}^{-\gamma_i}\gamma_i\Phi_i^{\gamma_i-1}\Phi_i' \\
&= 1 - \rho_i\Phi_i \\
&+ \left(1-\gamma_i\right)\left[\left(\frac{1}{\gamma_i\sigma_t}\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t} + \gamma_i y_t\sigma_{yt}\frac{\Phi_i'(y_t)}{\Phi_i(y_t)}\right)\sigma_t\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right) + r_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}\right) - \frac{1}{\Phi_i}\right]\Phi_i \\
&+ y_t\mu_{yt}^i\gamma_i\Phi_i' + \frac{1}{2}\Bigg\{-\left(\frac{1}{\gamma_i\sigma_t}\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t} + \gamma_i y_t\sigma_{yt}\frac{\Phi_i'(y_t)}{\Phi_i(y_t)}\right)\right)^2\sigma_t^2\gamma_i\left(1-\gamma_i\right)\Phi_i \\
&+ y_t^2\sigma_{yt}^2\gamma_i\left[\left(\gamma_i-1\right)\Phi_i^{-1}\left(\Phi_i'\right)^2 + \Phi_i''\right]\Bigg\} \\
&+ \left(1-\gamma_i\right)\frac{1}{\gamma_i\sigma_t}\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t} + \gamma_i y_t\sigma_{yt}\frac{\Phi_i'(y_t)}{\Phi_i(y_t)}\right)y_t\sigma_{yt}\sigma_t\gamma_i\Phi_i' \\
&= \frac{y_t^2\sigma_{yt}^2\gamma_i}{2}\Phi_i'' + \Bigg[\left(1-\gamma_i\right)y_t\sigma_{yt}\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right) + y_t\mu_{yt}^i\gamma_i - \frac{y_t^2\sigma_{yt}^2\Phi_i^{-1}\Phi_i'\gamma_i\left(1-\gamma_i\right)}{2} \\
&- \left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right)y_t\sigma_{yt}\left(1-\gamma_i\right) + \frac{\gamma_i\left(\gamma_i-1\right)y_t^2\sigma_{yt}^2\Phi_i^{-1}\Phi_i'}{2} + \left(1-\gamma_i\right)\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t} + \gamma_i y_t\sigma_{yt}\frac{\Phi_i'(y_t)}{\Phi_i(y_t)}\right)y_t\sigma_{yt}\Bigg]\Phi_i' \\
&+ \Bigg\{-\rho_i + \left(1-\gamma_i\right)\left(\frac{1}{\gamma_i\sigma_t}\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right)\sigma_t\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right) + r_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}\right) \\
&- \frac{1}{2}\left(\frac{1}{\gamma_i\sigma_t}\right)^2\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right)^2\sigma_t^2\gamma_i\left(1-\gamma_i\right)\Bigg\}\Phi_i \\
&+ 1 - 1 + \gamma_i \\
&= \frac{y_t^2\sigma_{yt}^2}{2}\Phi_i'' + y_t\left[\frac{\left(1-\gamma_i\right)}{\gamma_i}\sigma_{yt}\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right) + \mu_{yt}^i\right]\Phi_i' \\
&+ \left[-\rho_i + \left(1-\gamma_i\right)\left(\frac{1}{2\gamma_i}\left(\kappa_t^i + \frac{\underline{\nu}_{it}^* - \bar{\nu}_{it}^*\sigma_t}{\sigma_t}\right)^2 + r_t - \underline{\nu}_{it}^*\underline{\theta} + \bar{\nu}_{it}^*\bar{m}\right)\right]\frac{\Phi_i}{\gamma_i} + 1.
\end{aligned}$$

Next, we show that $\underline{\nu}_{it}^*, \bar{\nu}_{it}^*$, and σ_t satisfy a system of equations and can be expressed as functions of Φ_i, Ψ_i, and their derivatives. The short sale constraint $\theta_{it} \geq \underline{\theta}$ with $\underline{\theta} \leq 0$ and the Kuhn-Tucker condition imply that $\underline{\nu}^*_{it}\underline{\theta} \leq 0$. Using the optimality condition for θ^*_{it} (Equation (3.8)):

$$\theta^*_{it} = \frac{1}{\gamma_i \sigma_t}\left[\left(\kappa^i_t + \frac{\underline{\nu}^*_{it} - \bar{\nu}^*_{it}\sigma_t}{\sigma_t}\right) + \gamma_i y_t \sigma_{yt}\frac{\Phi'_i(y_t)}{\Phi_i(y_t)}\right] \geq \underline{\theta}$$

$$\implies \underline{\nu}^*_{it}\underline{\theta} \leq \gamma_i\left(\underline{\theta}\sigma_t\right)^2\left[1 - \frac{1}{\underline{\theta}\sigma_t}\left(\frac{\kappa^i_t - \bar{\nu}^*_{it}}{\gamma_i} + y_t\sigma_{yt}\frac{\Phi'_i}{\Phi_i}\right)\right].$$

Moreover, either the above inequality or $\underline{\nu}^*_{it}\underline{\theta} \leq 0$ holds as equality, depending on whether the constraint is binding. Therefore

$$\underline{\nu}^*_{it}\underline{\theta} = \min\left\{0, 1 - \frac{1}{\underline{\theta}\sigma_t}\left(\frac{\kappa^i_t - \bar{\nu}^*_{it}}{\gamma_i} + y_t\sigma_{yt}\frac{\Phi'_i}{\Phi_i}\right)\right\}\gamma_i\left(\underline{\theta}\sigma_t\right)^2.$$

Similarly, using the constraint

$$\theta^*_{it} = \frac{1}{\gamma_i \sigma_t}\left[\left(\kappa^i_t + \frac{\underline{\nu}^*_{it} - \bar{\nu}^*_{it}\sigma_t}{\sigma_t}\right) + \gamma_i y_t \sigma_{yt}\frac{\Phi'_i(y_t)}{\Phi_i(y_t)}\right] \leq \frac{\bar{m}}{\sigma_t}$$

and the Kuhn-Tucker condition $\bar{\nu}^*_{it}\bar{m}/\sigma_t \geq 0$, we have

$$\bar{\nu}^*_{it}\bar{m} = \max\left\{0, \frac{1}{\bar{m}}\left(\frac{\kappa^i_t + \underline{\nu}^*_{it}/\sigma_t}{\gamma_i} + y_t\sigma_{yt}\frac{\Phi'_i}{\Phi_i}\right) - 1\right\}\gamma_i\bar{m}^2.$$

Substituting κ^i_t and σ_{yt} from Equations (3.2) and (3.4), using the fact that $\kappa^i_t = \kappa_t + \left(\mu^i_D - \mu_D\right)/\sigma_D$:

$$\begin{aligned}
\underline{\nu}^*_{At}\underline{\theta} = \min\Bigg\{0, 1 - \frac{1}{\underline{\theta}\sigma_t}\Bigg(&\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D - \Gamma_t\frac{1-y_t}{\gamma_A}\frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \Gamma_t\frac{y_t}{\gamma_B}\frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} - \bar{\nu}^*_{At}}{\gamma_A} \\
&+ y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A - \gamma_B\right)\sigma_D + \Delta_D - \left(\frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t}\right)\right]\frac{\Phi'_A}{\Phi_A}\Bigg)\Bigg\}\gamma_A\left(\underline{\theta}\sigma_t\right)^2 \\
= \min\Bigg\{0, 1 - &\left[\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D}{\gamma_A} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left(\left(\gamma_A - \gamma_B\right)\sigma_D + \Delta_D\right)\frac{\Phi'_A}{\Phi_A} - \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(1 - \left(1 - y_t\right)\frac{\Phi'_A}{\Phi_A}\right)\left(\bar{\nu}^*_{At} - \bar{\nu}^*_{Bt}\right)\right]\frac{1}{\underline{\theta}\sigma_t} \\
&+ \left[\frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left[\frac{\gamma_B}{\gamma_A}\frac{1-y_t}{y_t} + \left(1 - y_t\right)\frac{\Phi'_A}{\Phi_A}\right]\underline{\nu}^*_{At} + \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left[1 - \left(1 - y_t\right)\frac{\Phi'_A}{\Phi_A}\right]\underline{\nu}^*_{Bt}\right]\frac{1}{\underline{\theta}\sigma^2_t}\Bigg\}\gamma_A\left(\underline{\theta}\sigma_t\right)^2.
\end{aligned}$$

$$\begin{aligned}
\underline{\nu}^*_{Bt}\underline{\theta} = \min\Bigg\{0, 1 - \frac{1}{\underline{\theta}\sigma_t}\Bigg(&\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D - \Gamma_t\frac{1-y_t}{\gamma_A}\frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \Gamma_t\frac{y_t}{\gamma_B}\frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t} - \bar{\nu}^*_{Bt}}{\gamma_B} \\
&+ y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A - \gamma_B\right)\sigma_D + \Delta_D - \left(\frac{\underline{\nu}^*_{At} - \bar{\nu}^*_{At}\sigma_t}{\sigma_t} - \frac{\underline{\nu}^*_{Bt} - \bar{\nu}^*_{Bt}\sigma_t}{\sigma_t}\right)\right]\frac{\Phi'_B}{\Phi_B}\Bigg)\Bigg\}\gamma_B\left(\underline{\theta}\sigma_t\right)^2 \\
= \min\Bigg\{0, 1 - &\left[\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D}{\gamma_B} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A - \gamma_B\right)\sigma_D + \Delta_D\right]\frac{\Phi'_B}{\Phi_B} + \frac{\Gamma_t\left(1 - y_t\right)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi'_B}{\Phi_B}\right)\left(\bar{\nu}^*_{At} - \bar{\nu}^*_{Bt}\right)\right]\frac{1}{\underline{\theta}\sigma_t} \\
&+ \left[\Gamma_t\frac{\left(1 - y_t\right)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi'_B}{\Phi_B}\right)\underline{\nu}^*_{At} + \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left[\frac{\gamma_A}{\gamma_B} - \left(1 - y_t\right)\frac{\Phi'_B}{\Phi_B}\right]\underline{\nu}^*_{Bt}\right]\frac{1}{\underline{\theta}\sigma^2_t}\Bigg\}\gamma_B\left(\underline{\theta}\sigma_t\right)^2
\end{aligned}$$

$$\bar{\nu}_{At}^{*}\bar{m} = \max\left\{0, \frac{1}{\bar{m}}\left(\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D - \Gamma_t\frac{1-y_t}{\gamma_A}\frac{\underline{\nu}_{At}^{*}-\bar{\nu}_{At}^{*}\sigma_t}{\sigma_t} - \Gamma_t\frac{y_t}{\gamma_B}\frac{\underline{\nu}_{Bt}^{*}-\bar{\nu}_{Bt}^{*}\sigma_t}{\sigma_t} + \underline{\nu}_{At}^{*}/\sigma_t}{\gamma_A}\right.\right.$$

$$\left.\left. + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A-\gamma_B)\sigma_D + \Delta_D - \left(\frac{\underline{\nu}_{At}^{*}-\bar{\nu}_{At}^{*}\sigma_t}{\sigma_t} - \frac{\underline{\nu}_{Bt}^{*}-\bar{\nu}_{Bt}^{*}\sigma_t}{\sigma_t}\right)\right]\frac{\Phi_A'}{\Phi_A}\right) - 1\right\}\gamma_A\bar{m}^2$$

$$= \max\left\{0, \frac{1}{\bar{m}}\left[\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D}{\gamma_A} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A-\gamma_B)\sigma_D + \Delta_D\right]\frac{\Phi_A'}{\Phi_A}\right.\right.$$

$$\left. + \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(\left[\frac{\gamma_B}{\gamma_A} + y_t\frac{\Phi_A'}{\Phi_A}\right]\bar{\nu}_{At}^{*} + \left[\frac{y_t}{1-y_t} - y_t\frac{\Phi_A'}{\Phi_A}\right]\bar{\nu}_{Bt}^{*}\right)\right]$$

$$\left. - \frac{1}{\bar{m}\sigma_t}\frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(-1 + (1-y_t)\frac{\Phi_A'}{\Phi_A}\right)(\underline{\nu}_{At}^{*} - \underline{\nu}_{Bt}^{*}) - 1\right\}\gamma_A\bar{m}^2.$$

$$\bar{\nu}_{Bt}^{*}\bar{m} = \max\left\{0, \frac{1}{\bar{m}}\left(\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D - \Gamma_t\frac{1-y_t}{\gamma_A}\frac{\underline{\nu}_{At}^{*}-\bar{\nu}_{At}^{*}\sigma_t}{\sigma_t} - \Gamma_t\frac{y_t}{\gamma_B}\frac{\underline{\nu}_{Bt}^{*}-\bar{\nu}_{Bt}^{*}\sigma_t}{\sigma_t} + \underline{\nu}_{Bt}^{*}/\sigma_t}{\gamma_B}\right.\right.$$

$$\left.\left. + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A-\gamma_B)\sigma_D + \Delta_D - \left(\frac{\underline{\nu}_{At}^{*}-\bar{\nu}_{At}^{*}\sigma_t}{\sigma_t} - \frac{\underline{\nu}_{Bt}^{*}-\bar{\nu}_{Bt}^{*}\sigma_t}{\sigma_t}\right)\right]\frac{\Phi_B'}{\Phi_B}\right) - 1\right\}\gamma_B\bar{m}^2$$

$$= \max\left\{0, \frac{1}{\bar{m}}\left[\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D}{\gamma_B} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A-\gamma_B)\sigma_D + \Delta_D\right]\frac{\Phi_B'}{\Phi_B}\right.\right.$$

$$\left. + \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(\left[1 + y_t\frac{\Phi_B'}{\Phi_B}\right]\bar{\nu}_{At}^{*} + \left[\frac{\gamma_A}{\gamma_B}\frac{y_t}{1-y_t} - y_t\frac{\Phi_B'}{\Phi_B}\right]\bar{\nu}_{Bt}^{*}\right)\right]$$

$$\left. - \frac{1}{\bar{m}\sigma_t}\frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi_B'}{\Phi_B}\right)(\underline{\nu}_{At}^{*} - \underline{\nu}_{Bt}^{*}) - 1\right\}\gamma_B\bar{m}^2.$$

To deal with the min/max operators in the multipliers, we could compare the unconstrained optimal portfolio weight θ_{it}^{u*} with the constraints. When $\underline{\theta} \leq \theta_{it}^{u*} \leq \bar{m}/\sigma_t$, $\underline{\nu}_{it}^{*} = \bar{\nu}_{it}^{*} = 0$. Otherwise, the non-zero expression for $\underline{\nu}_{it}^{*}$ or $\bar{\nu}_{it}^{*}$ applies when the corresponding constraint is violated. Finally, the exact values for the multipliers can be solved from the system of four equations, as functions of σ_t, Φ_i, Ψ_i, and their derivatives.

Note that the lower bound $\theta_{it}^{*} \geq \underline{\theta}$ can't be binding for both investors in order to clear the stock market, and that the lower and the upper bounds can't be both binding for a given investor. Therefore there are at most eight possible cases, as listed in Table 1.

$$\underline{\nu}_{At}^{*}\underline{\theta} = \min\left\{0, 1 - \left[\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D}{\gamma_A} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left((\gamma_A-\gamma_B)\sigma_D + \Delta_D\right)\frac{\Phi_A'}{\Phi_A}\right.\right.$$

$$\left. - \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(1 - (1-y_t)\frac{\Phi_A'}{\Phi_A}\right)(\bar{\nu}_{At}^{*} - \bar{\nu}_{Bt}^{*})\right]\frac{1}{\underline{\theta}\sigma_t}$$

$$\left. + \left[\frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left[\frac{\gamma_B}{\gamma_A} + y_t\frac{\Phi_A'}{\Phi_A}\right]\underline{\nu}_{At}^{*} + \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left[1 - (1-y_t)\frac{\Phi_A'}{\Phi_A}\right]\underline{\nu}_{Bt}^{*}\right]\frac{1}{\underline{\theta}\sigma_t^2}\right\}\gamma_A(\underline{\theta}\sigma_t)^2.$$

Table 1: Cases for binding constraints

investor / cases	lower		upper	
	A	B	A	B
1	n	n	n	n
2	b	n	n	n
3	n	b	n	n
4	n	n	b	n
5	n	n	n	b
6	b	n	n	b
7	n	b	b	n
8	n	n	b	b

Note: "n" denotes "non-binding", and "b" denotes "binding".

$$\underline{\nu}^*_{Bt}\underline{\theta} = \min\Bigg\{0, 1 - \Bigg[\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D}{\gamma_B} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A - \gamma_B)\sigma_D + \Delta_D\right]\frac{\Phi'_B}{\Phi_B}$$

$$+ \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi'_B}{\Phi_B}\right)(\bar{\nu}^*_{At} - \bar{\nu}^*_{Bt})\Bigg]\frac{1}{\underline{\theta}\sigma_t}$$

$$+ \left[\Gamma_t\frac{(1-y_t)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi'_B}{\Phi_B}\right)\underline{\nu}^*_{At} + \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left[\frac{\gamma_A}{\gamma_B} - (1-y_t)\frac{\Phi'_B}{\Phi_B}\right]\underline{\nu}^*_{Bt}\right]\frac{1}{\underline{\theta}\sigma_t^2}\Bigg\}\gamma_B(\underline{\theta}\sigma_t)^2$$

$$\bar{\nu}^*_{At}\bar{m} = \max\Bigg\{0, \frac{1}{\bar{m}}\Bigg[\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D}{\gamma_A} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A - \gamma_B)\sigma_D + \Delta_D\right]\frac{\Phi'_A}{\Phi_A}$$

$$+ \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(\left[\frac{\gamma_B}{\gamma_A} + y_t\frac{\Phi'_A}{\Phi_A}\right]\bar{\nu}^*_{At} + \left[\frac{y_t}{1-y_t} - y_t\frac{\Phi'_A}{\Phi_A}\right]\bar{\nu}^*_{Bt}\right)\Bigg]$$

$$- \frac{1}{\bar{m}\sigma_t}\frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(-1 + (1-y_t)\frac{\Phi'_A}{\Phi_A}\right)(\underline{\nu}^*_{At} - \underline{\nu}^*_{Bt}) - 1\Bigg\}\gamma_A\bar{m}^2$$

$$\bar{\nu}^*_{Bt}\bar{m} = \max\Bigg\{0, \frac{1}{\bar{m}}\Bigg[\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D}{\gamma_B} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A - \gamma_B)\sigma_D + \Delta_D\right]\frac{\Phi'_B}{\Phi_B}$$

$$+ \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(\left[1 + y_t\frac{\Phi'_B}{\Phi_B}\right]\bar{\nu}^*_{At} + \left[\frac{\gamma_A}{\gamma_B}\frac{y_t}{1-y_t} - y_t\frac{\Phi'_B}{\Phi_B}\right]\bar{\nu}^*_{Bt}\right)\Bigg]$$

$$- \frac{1}{\bar{m}\sigma_t}\frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi'_B}{\Phi_B}\right)(\underline{\nu}^*_{At} - \underline{\nu}^*_{Bt}) - 1\Bigg\}\gamma_B\bar{m}^2.$$

Case 1. No constraint binds. $\underline{\nu}^*_{At} = \underline{\nu}^*_{Bt} = \bar{\nu}^*_{At} = \bar{\nu}^*_{Bt} = 0$.

Case 2. Lower bound binds for investor A. $\underline{\nu}^*_{Bt} = \bar{\nu}^*_{At} = \bar{\nu}^*_{Bt} = 0$.

$$\begin{aligned}
\underline{\nu}^*_{At}\underline{\theta} &= \left\{1 - \left[\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D}{\gamma_A} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left(\left(\gamma_A-\gamma_B\right)\sigma_D + \Delta_D\right)\frac{\Phi_A'}{\Phi_A}\right]\frac{1}{\underline{\theta}\sigma_t}\right. \\
&\quad \left. + \left[\frac{\Gamma_t\left(1-y_t\right)}{\gamma_A\gamma_B}\left[\frac{\gamma_B}{\gamma_A} + y_t\frac{\Phi_A'}{\Phi_A}\right]\underline{\nu}^*_{At}\right]\frac{1}{\underline{\theta}\sigma_t^2}\right\}\gamma_A\left(\underline{\theta}\sigma_t\right)^2 \\
&= \left\{1 - \frac{g^A_{3t}}{\underline{\theta}\sigma_t} + \left(g^A_{2t} + \frac{1}{\gamma_A}\right)\frac{\underline{\nu}^*_{At}\underline{\theta}}{\left(\underline{\theta}\sigma_t\right)^2}\right\}\gamma_A\left(\underline{\theta}\sigma_t\right)^2
\end{aligned}$$

where

$$g^A_{2t} = \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(-1 + \left(1-y_t\right)\frac{\Phi_A'}{\Phi_A}\right), g^A_{3t} = \sigma_D + \left(\left(\gamma_A-\gamma_B\right)\sigma_D + \Delta_D\right)g^A_{2t}.$$

Rearranging,

$$\underline{\nu}^*_{At}\underline{\theta} = -\left(1 - \frac{g^A_{3t}}{\underline{\theta}\sigma_t}\right)\frac{\left(\underline{\theta}\sigma_t\right)^2}{g^A_{2t}}.$$

Finally,

$$\begin{aligned}
\sigma_t &= \frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A-\gamma_B\right)\sigma_D + \Delta_D - \frac{\underline{\nu}^*_{At}}{\sigma_t}\right] + \sigma_D \\
&= \frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A-\gamma_B\right)\sigma_D + \Delta_D + \left(1 - \frac{g^A_{3t}}{\underline{\theta}\sigma_t}\right)\frac{\underline{\theta}\sigma_t}{g^A_{2t}}\right] + \sigma_D \\
&= \frac{\left[\frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left(\gamma_A-\gamma_B\right)+1\right]\sigma_D + \frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\Delta_D - \frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\frac{g^A_{3t}}{g^A_{2t}}}{1 - \frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\frac{\underline{\theta}}{g^A_{2t}}} \\
&= \frac{\left[\frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left(\gamma_A-\gamma_B\right)+1\right]\sigma_D + \frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left(-\frac{\sigma_D}{g^A_{2t}} - \left(\gamma_A-\gamma_B\right)\sigma_D\right)}{1 - \frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\frac{\underline{\theta}}{g^A_{2t}}} \\
&= \frac{\left[\frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left(\gamma_A-\gamma_B\right)+1 - \frac{\Psi'}{\Psi}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left(\frac{1}{g^A_{2t}} + \left(\gamma_A-\gamma_B\right)\right)\right]\sigma_D}{1 - \frac{\Psi'(y_t)}{\Psi(y_t)}y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\frac{\underline{\theta}}{g^A_{2t}}} \\
&= \frac{\left[1 - \frac{\Psi'}{\Psi}\frac{1-y_t}{\left(-1+\left(1-y_t\right)\frac{\Phi_A'}{\Phi_A}\right)}\right]\sigma_D}{1 - \frac{\Psi'}{\Psi}\frac{\left(1-y_t\right)\underline{\theta}}{\left(-1+\left(1-y_t\right)\frac{\Phi_A'}{\Phi_A}\right)}} \\
&= \frac{\sigma_D}{\left(1-\underline{\theta}\right)g^A_{1t} + \underline{\theta}}
\end{aligned}$$

where

$$g^A_{1t} = \frac{-1 + \left(1-y_t\right)\frac{\Phi_A'}{\Phi_A}}{-1 + \left(1-y_t\right)\left(\frac{\Phi_A'}{\Phi_A} - \frac{\Psi'}{\Psi}\right)}.$$

Substituting for σ_t,

$$\begin{aligned}
\underline{\nu}^*_{At} &= -\frac{1}{\underline{\theta}}\left(1-\frac{g^A_{3t}}{\underline{\theta}\frac{\sigma_D}{(1-\underline{\theta})g^A_{1t}+\underline{\theta}}}\right)\frac{\left(\underline{\theta}\frac{\sigma_D}{(1-\underline{\theta})g^A_{1t}+\underline{\theta}}\right)^2}{g^A_{2t}} \\
&= \frac{\sigma_D\left[\sigma_D/g^A_{2t}+\left(\left(\gamma_A-\gamma_B\right)\sigma_D+\Delta_D\right)\right]}{\left(1-\underline{\theta}\right)g^A_{1t}+\underline{\theta}} - \frac{\underline{\theta}\sigma_D^2}{g^A_{2t}\left(\left(1-\underline{\theta}\right)g^A_{1t}+\underline{\theta}\right)^2} \\
&= \frac{\sigma_D\left[\left(\gamma_A-\gamma_B\right)\sigma_D+\Delta_D\right]}{\left(1-\underline{\theta}\right)g^A_{1t}+\underline{\theta}} + \frac{\left(1-\underline{\theta}\right)\sigma_D^2 g^A_{1t}}{g^A_{2t}\left(\left(1-\underline{\theta}\right)g^A_{1t}+\underline{\theta}\right)^2}.
\end{aligned}$$

Case 3. Lower bound binds for investor B. $\underline{\nu}^*_{At}=\bar{\nu}^*_{At}=\bar{\nu}^*_{Bt}=0$.

$$\begin{aligned}
\underline{\nu}^*_{Bt}\underline{\theta} &= \left\{1-\left[\frac{\Gamma_t\sigma_D+\Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D}{\gamma_B}+y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A-\gamma_B\right)\sigma_D+\Delta_D\right]\frac{\Phi_B'}{\Phi_B}\right]\frac{1}{\underline{\theta}\sigma_t}\right. \\
&\quad \left. +\left[\frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(\frac{\gamma_A}{\gamma_B}-\left(1-y_t\right)\frac{\Phi_B'}{\Phi_B}\right)\underline{\nu}^*_{Bt}\right]\frac{1}{\underline{\theta}\sigma_t^2}\right\}\gamma_B\left(\underline{\theta}\sigma_t\right)^2 \\
&= \left\{1-\frac{g^B_{3t}}{\underline{\theta}\sigma_t}-\left(g^B_{2t}-\frac{1}{\gamma_B}\right)\frac{\underline{\nu}^*_{Bt}\underline{\theta}}{\left(\underline{\theta}\sigma_t\right)^2}\right\}\gamma_B\left(\underline{\theta}\sigma_t\right)^2
\end{aligned}$$

where

$$g^B_{2t}=\frac{\Gamma_t\left(1-y_t\right)}{\gamma_A\gamma_B}\left(1+y_t\frac{\Phi_B'}{\Phi_B}\right), g^B_{3t}=\sigma_D+\left(\left(\gamma_A-\gamma_B\right)\sigma_D+\Delta_D\right)g^B_{2t}.$$

Rearranging,

$$\underline{\nu}^*_{Bt}\underline{\theta}=\left(1-\frac{g^B_{3t}}{\underline{\theta}\sigma_t}\right)\frac{\left(\underline{\theta}\sigma_t\right)^2}{g^B_{2t}}.$$

Finally,

$$\begin{aligned}
\sigma_t &= \frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B)\, \sigma_D + \Delta_D + \frac{\underline{\nu}^*_{Bt}}{\sigma_t} \right] + \sigma_D \\
&= \frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B)\, \sigma_D + \Delta_D + \left(1 - \frac{g^B_{3t}}{\underline{\theta}\sigma_t} \right) \frac{\underline{\theta}\sigma_t}{g^B_{2t}} \right] + \sigma_D \\
&= \frac{\left[\frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} (\gamma_A - \gamma_B) + 1 \right] \sigma_D + \frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} \Delta_D - \frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} \frac{g^B_{3t}}{g^B_{2t}}}{1 - \frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} \frac{\underline{\theta}}{g^B_{2t}}} \\
&= \frac{\left[\frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} (\gamma_A - \gamma_B) + 1 \right] \sigma_D + \frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} \left(-\frac{\sigma_D}{g^B_{2t}} - (\gamma_A - \gamma_B)\, \sigma_D \right)}{1 - \frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} \frac{\underline{\theta}}{g^B_{2t}}} \\
&= \frac{\left[\frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} (\gamma_A - \gamma_B) + 1 - \frac{\Psi'}{\Psi} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} \left(\frac{1}{g^B_{2t}} + (\gamma_A - \gamma_B) \right) \right] \sigma_D}{1 - \frac{\Psi'(y_t)}{\Psi(y_t)} y_t \Gamma_t \frac{1-y_t}{\gamma_A\gamma_B} \frac{\underline{\theta}}{g^B_{2t}}} \\
&= \frac{\left[1 - \frac{\Psi'}{\Psi} \frac{y_t}{\left(1 + y_t \frac{\Phi'_B}{\Phi_B}\right)} \right] \sigma_D}{1 - \frac{\Psi'}{\Psi} \frac{y_t \underline{\theta}}{\left(1 + y_t \frac{\Phi'_B}{\Phi_B}\right)}} \\
&= \frac{\sigma_D}{(1-\underline{\theta})\, g^B_{1t} + \underline{\theta}}
\end{aligned}$$

where

$$g^B_{1t} = \frac{1 + y_t \frac{\Phi'_B}{\Phi_B}}{1 + y_t \left(\frac{\Phi'_B}{\Phi_B} - \frac{\Psi'}{\Psi} \right)}.$$

Substituting for σ_t,

$$\begin{aligned}
\underline{\nu}^*_{Bt} &= \frac{1}{\underline{\theta}} \left(1 - \frac{g^B_{3t}}{\underline{\theta} \frac{\sigma_D}{(1-\underline{\theta}) g^B_{1t} + \underline{\theta}}} \right) \frac{\left(\underline{\theta} \frac{\sigma_D}{(1-\underline{\theta}) g^B_{1t} + \underline{\theta}} \right)^2}{g^B_{2t}} \\
&= \frac{\underline{\theta}\sigma_D^2}{g^B_{2t} \left((1-\underline{\theta})\, g^B_{1t} + \underline{\theta} \right)^2} - \frac{\sigma_D \left[\sigma_D / g^B_{2t} + ((\gamma_A - \gamma_B)\, \sigma_D + \Delta_D) \right]}{(1-\underline{\theta})\, g^B_{1t} + \underline{\theta}} \\
&= -\frac{\sigma_D \left[(\gamma_A - \gamma_B)\, \sigma_D + \Delta_D \right]}{(1-\underline{\theta})\, g^A_{1t} + \underline{\theta}} - \frac{(1-\underline{\theta})\, \sigma_D^2 g^B_{1t}}{g^B_{2t} \left((1-\underline{\theta})\, g^B_{1t} + \underline{\theta} \right)^2}.
\end{aligned}$$

Case 4. Upper bound binds for investor A. $\underline{\nu}^*_{At} = \underline{\nu}^*_{Bt} = \bar{\nu}^*_{Bt} = 0$.

$$\bar{\nu}^*_{At}\bar{m} = \left\{ \frac{1}{\bar{m}} \left[\frac{\Gamma_t \sigma_D - \Gamma_t \frac{y_t}{\gamma_B} \Delta_D}{\gamma_A} + y_t \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B) \sigma_D + \Delta_D \right] \frac{\Phi'_A}{\Phi_A} \right.\right.$$
$$\left.\left. + \frac{\Gamma_t (1-y_t)}{\gamma_A \gamma_B} \left(\left[\frac{\gamma_B}{\gamma_A} + y_t \frac{\Phi'_A}{\Phi_A} \right] \bar{\nu}^*_{At} \right) \right] - 1 \right\} \gamma_A \bar{m}^2$$
$$= \left\{ \frac{1}{\bar{m}} \left[g^A_{3t} + \left(g^A_{2t} + \frac{1}{\gamma_A} \right) \bar{\nu}^*_{At} \right] - 1 \right\} \gamma_A \bar{m}^2.$$

Rearranging,

$$\bar{\nu}^*_{At} = \left(1 - \frac{g^A_{3t}}{\bar{m}} \right) \frac{\bar{m}}{g^A_{2t}}.$$

Substituting for $\bar{\nu}^*_{At}$ in the expression for σ_t:

$$\sigma_t = \frac{\Psi'(y_t)}{\Psi(y_t)} y_t \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B) \sigma_D + \Delta_D + \bar{\nu}^*_{At} \right] + \sigma_D$$
$$= \frac{\Psi'(y_t)}{\Psi(y_t)} y_t \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B) \sigma_D + \Delta_D + \left(1 - \frac{g^A_{3t}}{\bar{m}} \right) \frac{\bar{m}}{g^A_{2t}} \right] + \sigma_D.$$

Case 5. Upper bound binds for investor B. $\underline{\nu}^*_{At} = \bar{\nu}^*_{At} = \underline{\nu}^*_{Bt} = 0$.

$$\bar{\nu}^*_{Bt}\bar{m} = \left\{ \frac{1}{\bar{m}} \left[\frac{\Gamma_t \sigma_D + \Gamma_t \frac{1-y_t}{\gamma_A} \Delta_D}{\gamma_B} + y_t \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B) \sigma_D + \Delta_D \right] \frac{\Phi'_B}{\Phi_B} \right.\right.$$
$$\left.\left. + \frac{\Gamma_t (1-y_t)}{\gamma_A \gamma_B} \left(\frac{\gamma_A}{\gamma_B} \frac{y_t}{1-y_t} - y_t \frac{\Phi'_B}{\Phi_B} \right) \bar{\nu}^*_{Bt} \right] - 1 \right\} \gamma_B \bar{m}^2$$
$$= \left\{ \frac{1}{\bar{m}} \left[g^B_{3t} - \left(g^B_{2t} - \frac{1}{\gamma_B} \right) \bar{\nu}^*_{Bt} \right] - 1 \right\} \gamma_B \bar{m}^2.$$

Rearranging,

$$\bar{\nu}^*_{Bt} = \left(\frac{g^B_{3t}}{\bar{m}} - 1 \right) \frac{\bar{m}}{g^B_{2t}}.$$

Substituting for $\bar{\nu}^*_{Bt}$ in the expression for σ_t:

$$\sigma_t = \frac{\Psi'(y_t)}{\Psi(y_t)} y_t \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B) \sigma_D + \Delta_D - \bar{\nu}^*_{Bt} \right] + \sigma_D$$
$$= \frac{\Psi'(y_t)}{\Psi(y_t)} y_t \Gamma_t \frac{1-y_t}{\gamma_A \gamma_B} \left[(\gamma_A - \gamma_B) \sigma_D + \Delta_D - \left(\frac{g^B_{3t}}{\bar{m}} - 1 \right) \frac{\bar{m}}{g^B_{2t}} \right] + \sigma_D.$$

Case 6. Lower bound binds for investor A and upper bound binds for investor B. $\bar{\nu}^*_{At} = \underline{\nu}^*_{Bt} = 0$.

$$\begin{aligned}
\underline{\nu}^*_{At}\underline{\theta} =& \Bigg\{1 - \left[\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D}{\gamma_A} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left(\left(\gamma_A - \gamma_B\right)\sigma_D + \Delta_D\right)\frac{\Phi'_A}{\Phi_A}\right.\\
&\left. + \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(1 - \left(1 - y_t\right)\frac{\Phi'_A}{\Phi_A}\right)\bar{\nu}^*_{Bt}\right]\frac{1}{\underline{\theta}\sigma_t} + \left[\frac{\Gamma_t\left(1-y_t\right)}{\gamma_A\gamma_B}\left(\frac{\gamma_B}{\gamma_A} + y_t\frac{\Phi'_A}{\Phi_A}\right)\underline{\nu}^*_{At}\right]\frac{1}{\underline{\theta}\sigma_t^2}\Bigg\}\gamma_A\left(\underline{\theta}\sigma_t\right)^2\\
=& \left\{1 - \left[g^A_{3t} - g^A_{2t}\bar{\nu}^*_{Bt}\right]\frac{1}{\underline{\theta}\sigma_t} + \left(g^A_{2t} + \frac{1}{\gamma_A}\right)\frac{\underline{\nu}^*_{At}}{\underline{\theta}\sigma_t^2}\right\}\gamma_A\left(\underline{\theta}\sigma_t\right)^2.
\end{aligned}$$

$$\begin{aligned}
\bar{\nu}^*_{Bt}\bar{m} =& \Bigg\{\frac{1}{\bar{m}}\left[\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D}{\gamma_B} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A - \gamma_B\right)\sigma_D + \Delta_D\right]\frac{\Phi'_B}{\Phi_B}\right.\\
&\left. + \frac{\Gamma_t\left(1-y_t\right)}{\gamma_A\gamma_B}\left(\frac{\gamma_A}{\gamma_B}\frac{y_t}{1-y_t} - y_t\frac{\Phi'_B}{\Phi_B}\right)\bar{\nu}^*_{Bt}\right] - \frac{1}{\bar{m}\sigma_t}\frac{\Gamma_t\left(1-y_t\right)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi'_B}{\Phi_B}\right)\underline{\nu}^*_{At} - 1\Bigg\}\gamma_B\bar{m}^2\\
=& \left\{\frac{1}{\bar{m}}\left[g^B_{3t} - \left(g^B_{2t} - \frac{1}{\gamma_B}\right)\bar{\nu}^*_{Bt}\right] - \frac{1}{\bar{m}\sigma_t}g^B_{2t}\underline{\nu}^*_{At} - 1\right\}\gamma_B\bar{m}^2.
\end{aligned}$$

This is a system of linear equations in two unknowns $(\underline{\nu}^*_{At}, \bar{\nu}^*_{Bt})$,which simplifies to

$$\begin{aligned}
\underline{\nu}^*_{At} + \sigma_t\bar{\nu}^*_{Bt} &= \left(\frac{g^A_{3t}}{\underline{\theta}\sigma_t} - 1\right)\frac{\underline{\theta}\sigma_t^2}{g^A_{2t}}\\
\frac{1}{\sigma_t}\underline{\nu}^*_{At} + \bar{\nu}^*_{Bt} &= \left(\frac{g^B_{3t}}{\bar{m}} - 1\right)\frac{\bar{m}}{g^B_{2t}}.
\end{aligned}$$

The system has no solution because the determinant of the coefficient matrix on the left side is zero.

Case 7. Lower bound binds for investor B and upper bound binds for investor A. $\underline{\nu}^*_{At} = \bar{\nu}^*_{Bt} = 0$.

$$\begin{aligned}
\bar{\nu}^*_{At}\bar{m} =& \Bigg\{\frac{1}{\bar{m}}\left[\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D}{\gamma_A} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A - \gamma_B\right)\sigma_D + \Delta_D\right]\frac{\Phi'_A}{\Phi_A} + \frac{\Gamma_t\left(1-y_t\right)}{\gamma_A\gamma_B}\left(\frac{\gamma_B}{\gamma_A} + y_t\frac{\Phi'_A}{\Phi_A}\right)\bar{\nu}^*_{At}\right]\\
&+ \frac{1}{\bar{m}\sigma_t}\frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left(-1 + \left(1-y_t\right)\frac{\Phi'_A}{\Phi_A}\right)\underline{\nu}^*_{Bt} - 1\Bigg\}\gamma_A\bar{m}^2\\
=& \left\{\frac{1}{\bar{m}}\left[g^A_{3t} + \left(g^A_{2t} + \frac{1}{\gamma_A}\right)\bar{\nu}^*_{At}\right] + \frac{1}{\bar{m}\sigma_t}g^A_{2t}\underline{\nu}^*_{Bt} - 1\right\}\gamma_A\bar{m}^2,
\end{aligned}$$

$$\begin{aligned}
\underline{\nu}^*_{Bt}\underline{\theta} =& \Bigg\{1 - \left[\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D}{\gamma_B} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[\left(\gamma_A - \gamma_B\right)\sigma_D + \Delta_D\right]\frac{\Phi'_B}{\Phi_B} + \frac{\Gamma_t\left(1-y_t\right)}{\gamma_A\gamma_B}\left(1 + y_t\frac{\Phi'_B}{\Phi_B}\right)\bar{\nu}^*_{At}\right]\frac{1}{\underline{\theta}\sigma_t}\\
&+ \frac{\Gamma_t y_t}{\gamma_A\gamma_B}\left[\frac{\gamma_A}{\gamma_B} - \left(1-y_t\right)\frac{\Phi'_B}{\Phi_B}\right]\frac{\underline{\nu}^*_{Bt}}{\underline{\theta}\sigma_t^2}\Bigg\}\gamma_B\left(\underline{\theta}\sigma_t\right)^2\\
=& \left\{1 - \left[g^B_{3t} + g^B_{2t}\bar{\nu}^*_{At}\right]\frac{1}{\underline{\theta}\sigma_t} - \left(g^B_{2t} - \frac{1}{\gamma_B}\right)\frac{\underline{\nu}^*_{Bt}}{\underline{\theta}\sigma_t^2}\right\}\gamma_B\left(\underline{\theta}\sigma_t\right)^2.
\end{aligned}$$

The system simplifies to

$$\bar{\nu}_{At}^{*} + \frac{1}{\sigma_t}\underline{\nu}_{Bt}^{*} = \left(1 - \frac{g_{3t}^{A}}{\bar{m}}\right)\frac{\bar{m}}{g_{2t}^{A}},$$
$$\sigma_t\bar{\nu}_{At}^{*} + \underline{\nu}_{Bt}^{*} = \left(1 - \frac{g_{3t}^{B}}{\underline{\theta}\sigma_t}\right)\frac{\underline{\theta}\sigma_t^{2}}{g_{2t}^{B}}.$$

Again, the system has no solution.

Case 8. Upper bound binds for both investors. $\underline{\nu}_{At}^{*} = \underline{\nu}_{Bt}^{*} = 0$.

$$\begin{aligned}\bar{\nu}_{At}^{*}\bar{m} = &\left\{\frac{1}{\bar{m}}\left[\frac{\Gamma_t\sigma_D - \Gamma_t\frac{y_t}{\gamma_B}\Delta_D}{\gamma_A} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A-\gamma_B)\sigma_D + \Delta_D\right]\frac{\Phi_A'}{\Phi_A}\right.\right.\\ &\left.\left. + \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(\left[\frac{\gamma_B}{\gamma_A} + y_t\frac{\Phi_A'}{\Phi_A}\right]\bar{\nu}_{At}^{*} + \left[\frac{y_t}{1-y_t} - y_t\frac{\Phi_A'}{\Phi_A}\right]\bar{\nu}_{Bt}^{*}\right)\right] - 1\right\}\gamma_A\bar{m}^2\\ = &\left\{\frac{1}{\bar{m}}\left[g_{3t}^{A} + \left(g_{2t}^{A} + \frac{1}{\gamma_A}\right)\bar{\nu}_{At}^{*} - g_{2t}^{A}\bar{\nu}_{Bt}^{*}\right] - 1\right\}\gamma_A\bar{m}^2,\end{aligned}$$

$$\begin{aligned}\bar{\nu}_{Bt}^{*}\bar{m} = &\left\{\frac{1}{\bar{m}}\left[\frac{\Gamma_t\sigma_D + \Gamma_t\frac{1-y_t}{\gamma_A}\Delta_D}{\gamma_B} + y_t\Gamma_t\frac{1-y_t}{\gamma_A\gamma_B}\left[(\gamma_A-\gamma_B)\sigma_D + \Delta_D\right]\frac{\Phi_B'}{\Phi_B}\right.\right.\\ &\left.\left. + \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\left(\left[1 + y_t\frac{\Phi_B'}{\Phi_B}\right]\bar{\nu}_{At}^{*} + \left[\frac{\gamma_A}{\gamma_B}\frac{y_t}{1-y_t} - y_t\frac{\Phi_B'}{\Phi_B}\right]\bar{\nu}_{Bt}^{*}\right)\right] - 1\right\}\gamma_B\bar{m}^2\\ = &\left\{\frac{1}{\bar{m}}\left[g_{3t}^{B} + g_{2t}^{B}\bar{\nu}_{At}^{*} - \left(g_{2t}^{B} - \frac{1}{\gamma_B}\right)\bar{\nu}_{Bt}^{*}\right] - 1\right\}\gamma_B\bar{m}^2.\end{aligned}$$

The system simplifies to

$$\bar{\nu}_{At}^{*} - \bar{\nu}_{Bt}^{*} = \left(1 - \frac{g_{3t}^{A}}{\bar{m}}\right)\frac{\bar{m}}{g_{2t}^{A}},$$
$$-\bar{\nu}_{At}^{*} + \bar{\nu}_{Bt}^{*} = \left(\frac{g_{3t}^{B}}{\bar{m}} - 1\right)\frac{\bar{m}}{g_{2t}^{B}}.$$

Again, the system has no solution.

Numerical method